J. Math. Anal. Appl. 389 (2012) 705–712

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A Blaschke-type condition for analytic functions on finitely connected domains. Applications to complex perturbations of a finite-band selfadjoint operator

L. Golinskii[a], S. Kupin[b,c,∗,1]

[a] Mathematics Division, Institute for Low Temperature Physics and Engineering, 47 Lenin ave., 61103 Kharkov, Ukraine
[b] Université Bordeaux, IMB, UMR 5251, F-33405 Talence, France
[c] CNRS, IMB, UMR 5251, F-33405 Talence, France

ARTICLE INFO

Article history:
Received 9 September 2011
Available online 7 December 2011
Submitted by L. Fialkow

Keywords:
Blaschke-type estimates
Lieb–Thirring type inequalities
Finite-band selfadjoint operators
Complex compact perturbation

ABSTRACT

This is a sequel of the article by Borichev, Golinskii and Kupin (2009) [1], where the authors obtain Blaschke-type conditions for special classes of analytic functions in the unit disk, which satisfy certain growth hypotheses. These results were applied to get Lieb–Thirring inequalities for complex compact perturbations of a selfadjoint operator with a simply connected resolvent set. The first result of the present paper is an appropriate local version of the Blaschke-type condition from Borichev et al. (2009) [1]. We apply it to obtain a similar condition for an analytic function in a finitely connected domain of a special type. Such condition is by and large the same as a Lieb–Thirring type inequality for complex compact perturbations of a selfadjoint operator with a finite-band spectrum. A particular case of this result is the Lieb–Thirring inequality for a selfadjoint perturbation of the Schatten class of a periodic (or finite-band) Jacobi matrix.

0. Introduction

Let $e = \{\alpha_j, \beta_j\}_{j=1,\dots,n+1} \subset \mathbb{R}$ be a set of distinct points. We suppose that

$$-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_{n+1} < \beta_{n+1} < +\infty. \tag{0.1}$$

Let also

$$\mathfrak{e} = \bigcup_{j=1}^{n+1} \mathfrak{e}_j, \quad \mathfrak{e}_j = [\alpha_j, \beta_j], \tag{0.2}$$

and $\Omega := \overline{\mathbb{C}} \backslash \mathfrak{e}$. For a function f analytic in Ω, $f \in \mathcal{A}(\Omega)$, Z_f stands for the set of the zeros counting their multiplicities. By $\mathrm{d}(\lambda, M)$ we denote the distance between a point λ and a set M.

Our main functional theoretic result looks as follows.

∗ Corresponding author at: Université Bordeaux, IMB, UMR 5251, F-33405 Talence, France.
E-mail addresses: leonid.golinskii@gmail.com (L. Golinskii), skupin@math.u-bordeaux1.fr (S. Kupin).

1 The second author is partially supported by ANR grants ANR-07-BLAN-024701 and ANR-09-BLAN-005801.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.12.011

Theorem 0.1. *Let $f \in \mathcal{A}(\Omega)$, $|f(\infty)| = 1$, and, for $p, q \geqslant 0$*

$$\log\bigl|f(\lambda)\bigr| \leqslant \frac{K_1}{\mathrm{d}^p(\lambda, \mathfrak{e})\mathrm{d}^q(\lambda, e)}. \tag{0.3}$$

Then for any $0 < \varepsilon < 1$

$$\sum_{\lambda \in Z_f} \mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})\mathrm{d}^{a(p,q,\varepsilon)}(\lambda, e)\bigl(1 + |\lambda|\bigr)^{b(p,q,\varepsilon)} \leqslant C \cdot K_1, \tag{0.4}$$

where

$$a(p, q, \varepsilon) = \frac{(p + 2q - 1 + \varepsilon)_+ - (p + 1 + \varepsilon)}{2},$$
$$b(p, q, \varepsilon) = (p + q - 1 + \varepsilon)_+ - \frac{(p + 2q - 1 + \varepsilon)_+ + p + 1 + \varepsilon}{2}.$$

As usual, $x_+ = \max\{x, 0\}$. Here and in the sequel $C = C(\mathfrak{e}, p, q, \varepsilon)$ stands for a generic positive constant, which depends on indicated parameters. Note that due to the normalization at infinity the set Z_f is bounded, so

$$\sum_{\lambda \in Z_f} \mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})\mathrm{d}^{a(p,q,\varepsilon)}(\lambda, e) < \infty.$$

Of course, inequality (0.4) looks somewhat cumbersome, and it can be simplified in specific situations. Here are two examples.

Corollary 0.2. *Let $f \in \mathcal{A}(\Omega)$, $|f(\infty)| = 1$, and, for $p, q \geqslant 0$, $p + q \geqslant 1$*

$$\log\bigl|f(\lambda)\bigr| \leqslant \frac{K_1}{\mathrm{d}^p(\lambda, \mathfrak{e})\mathrm{d}^q(\lambda, e)}.$$

Then for any $0 < \varepsilon < 1$

$$\sum_{\lambda \in Z_f} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})\mathrm{d}^{q-1}(\lambda, e)}{1 + |\lambda|} \leqslant C \cdot K_1. \tag{0.5}$$

The case $q = 0$ is particularly important for applications.

Corollary 0.3. *Let $f \in \mathcal{A}(\Omega)$, $|f(\infty)| = 1$, and*

$$\log\bigl|f(\lambda)\bigr| \leqslant \frac{K_1}{\mathrm{d}^p(\lambda, \mathfrak{e})}, \quad p \geqslant 0. \tag{0.6}$$

Then for any $0 < \varepsilon < 1$

$$\sum_{\lambda \in Z_f} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})}{\mathrm{d}(\lambda, e)(1 + |\lambda|)} \leqslant C \cdot K_1, \tag{0.7}$$

as long as $p \geqslant 1$, and

$$\sum_{\lambda \in Z_f} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})}{(\mathrm{d}(\lambda, e)(1 + |\lambda|))^{(p+1+\varepsilon)/2}} \leqslant C \cdot K_1 \tag{0.8}$$

for $p < 1$.

All operators appearing in the present paper act on a separable Hilbert space H. Consider a (bounded) selfadjoint operator A_0 defined on H. We suppose it to be finite-band, i.e., for its spectrum

$$\sigma(A_0) = \sigma_{ess}(A_0) = \mathfrak{e}, \tag{0.9}$$

$\mathfrak{e}$ is defined in (0.2). A typical example here is a double infinite periodic Jacobi matrix.

Let $B \in \mathcal{S}_p$, the Schatten class of operators, $p > 0$. We do not suppose B to be selfadjoint. By the Weyl theorem (see, e.g., [15]) the essential spectrum $\sigma_{ess}(A)$, $A = A_0 + B$, coincides with $\sigma_{ess}(A_0)$.

We want to gather some information on the distribution of the discrete spectrum $\sigma_d(A) := \sigma(A) \backslash \sigma_{ess}(A)$, which consists of eigenvalues of finite algebraic multiplicity. It is clear that the points from $\sigma_d(A)$ can only accumulate to $\mathfrak{e}$. Here is the quantitative version of this intuition.

Theorem 0.4. *Let A_0 be as described above, $B \in \mathcal{S}_p$ and $A = A_0 + B$. Then, for $0 < \varepsilon < 1$ and $p \geqslant 1$*

$$\sum_{\lambda \in \sigma_d(A)} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})}{\mathrm{d}(\lambda, e)(1+|\lambda|)} \leqslant C \cdot \|B\|_{\mathcal{S}_p}^p, \tag{0.10}$$

and for $0 < p < 1$

$$\sum_{\lambda \in \sigma_d(A)} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})}{(\mathrm{d}(\lambda, e)(1+|\lambda|))^{(p+1+\varepsilon)/2}} \leqslant C \cdot \|B\|_{\mathcal{S}_p}^p. \tag{0.11}$$

Remark 0.5. The case $n = 0$, i.e., $\sigma(A_0) = [\alpha, \beta]$, is not exceptional. The point is that for $e = \{\alpha, \beta\}$

$$C_1\big|(\lambda - \alpha)(\lambda - \beta)\big| \leqslant \mathrm{d}(\lambda, e)\big(1 + |\lambda|\big) \leqslant C_2\big|(\lambda - \alpha)(\lambda - \beta)\big|, \quad \lambda \in \mathbb{C},$$

with absolute constants $C_{1,2}$, so we come to Theorem 2.3 from [1].

The case of *selfadjoint* perturbation B is well developed. A general result due to Kato [2] states that for A_0 with (0.9)

$$\sum_{\lambda \in \sigma_d(A)} \mathrm{d}^p(\lambda, \mathfrak{e}) \leqslant \|B\|_{\mathcal{S}_p}^p, \quad p \geqslant 1,$$

which is stronger than (0.10). For almost periodic Jacobi matrices with (0.9) the latter result (for $p = 1$) is sharpened in Hundertmark and Simon [3, Theorem 1.3]

$$\sum_{\lambda \in \sigma_d(A)} \mathrm{d}^{\frac{1}{2}+\varepsilon}(\lambda, \mathfrak{e}) \leqslant C\|B\|_{\mathcal{S}_1}, \quad \forall \varepsilon > 0.$$

For periodic A_0 the same result holds with $\varepsilon = 0$ [4].

A number of interesting results on Lieb–Thirring inequalities for nonselfadjoint compact perturbations of the discrete Laplacian are in [5–8]. A general operator theoretic approach is suggested in [9]. We thank Marcel Hansmann for informing us of this paper.

As usual, we write $\mathbb{D} = \{z\colon |z| < 1\}$ for the unit disk, $\mathbb{T} = \{z\colon |z| = 1\}$ for the unit circle, and $B(w_0, r) = \{w\colon |w - w_0| < r\}$ for balls in the complex plane. Sometimes, we label the balls by the variable of the corresponding complex plane, i.e. $B_w(z_0, r)$ ($B_\lambda(z_0, r)$) stays for a ball in the w-plane (the λ-plane), respectively.

1. Local version of Borichev–Golinskii–Kupin theorem

We begin with the result of Borichev, Golinskii and Kupin [1, Theorem 0.2] and its version in [8, Theorem 4].

Theorem 1.1. *Let $I = \{\zeta_j\}_{j=1}^k$ be a finite subset of $\mathbb{T}$, $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, and for $p', q', s \geqslant 0$*

$$\log\big|f(z)\big| \leqslant \frac{K|z|^s}{\mathrm{d}^{p'}(z, \mathbb{T})\mathrm{d}^{q'}(z, I)}, \quad z \in \mathbb{D}.$$

Then for any $0 < \varepsilon < 1$

$$\sum_{z \in Z_f} \frac{\mathrm{d}^{p'+1+\varepsilon}(z, \mathbb{T})}{|z|^{(s-1+\varepsilon)_+}} \mathrm{d}^{(q'-1+\varepsilon)_+}(z, I) \leqslant C\big(I, p', q', \varepsilon\big) \cdot K.$$

Our goal here is to prove a local version of the above result (cf. [10, Theorem 7]).

Let $G \subset \overline{\mathbb{D}}$ be an open circular polygon, $0 \in G$, with vertices $\{u_i\} \in \mathbb{T}$, and sides (arcs) $\tau_i = [u_i, u_{i+1}]$, $i = 1, 2, \ldots, 2N$, $u_{2N+1} = u_1$ (see Fig. 1). The arcs τ_{2j} lie on $\mathbb{T}$, and τ_{2j-1}, which we call the inner sides of G, lie on some orthocircles, that is, circles orthogonal to $\mathbb{T}$. Put

$$\Delta_1 = \partial G \cap \mathbb{D} = \left\{\bigcup_{j=1}^N (u_{2j-1}, u_{2j})\right\}, \qquad \Delta_2 = \partial G \cap \mathbb{T} = \left\{\bigcup_{j=1}^N [u_{2j}, u_{2j+1}]\right\},$$

so $\partial G = \Delta_1 \cup \Delta_2$.

Take a finite set $E = \{\zeta_j\}_{j=1}^k$ on the unit circle,

$$E \subset \Delta_2^o = \bigcup_{j=1}^N (u_{2j}, u_{2j+1}).$$

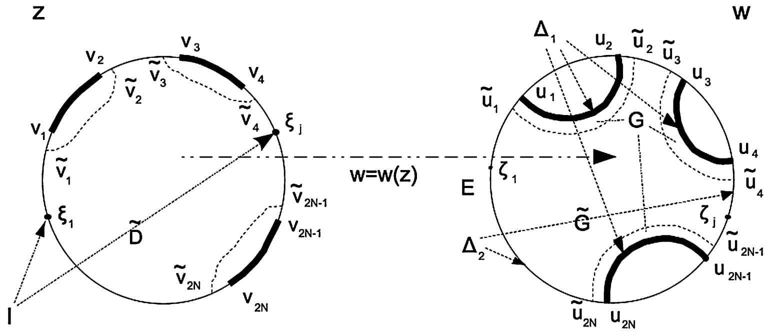

Fig. 1. The domains G, $\tilde{G}$ and the map w.

We set $\tilde{G} \subset G$ to be a properly "shrunk" circular polygon, in such a way that $E \subset \tilde{\Delta}_2^o$, see again Fig. 1. The notation for $\tilde{G}$ is the same as for G up to "waves" referring to the first set. So, for instance, the vertices of $\tilde{G}$ are $\tilde{u}_i$,

$$\partial\tilde{G} = \tilde{\Delta}_1 \cup \tilde{\Delta}_2, \quad \tilde{\Delta}_1 = \partial\tilde{G} \cap \mathbb{D}, \ \tilde{\Delta}_2 = \partial\tilde{G} \cap \mathbb{T}.$$

It is important that $\min_j \mathrm{d}(\tau_{2j-1}, \tilde{\tau}_{2j-1}) = \mathrm{d}' > 0$, $j = 1, 2, \ldots, N$.

Consider a conformal map w, $w : \mathbb{D} \to G$, normalized by $w(0) = 0$, $w'(0) > 0$. Sometimes, to indicate explicitly the variables, we will write $w : \mathbb{D}_z \to G_w$.

Put $\tilde{D} = w^{-1}(\tilde{G}) \subset \mathbb{D}_z$ and introduce

- preimages of vertices $v_j = w^{-1}(u_j)$, $\tilde{v}_j = w^{-1}(\tilde{u}_j)$, $j = 1, \ldots, 2N$,
- preimages of sides $\tilde{\tau}_j = w^{-1}(\tau_j) \subset \mathbb{T}_z$, $j = 1, \ldots, 2N$,
- preimages of selected points $I = \{\xi_j = w^{-1}(\zeta_j)\}$, $j = 1, \ldots, k$.

Clearly, I is contained in the closure of $\tilde{D}$.

For short, we write $w = w(z)$. Here is a couple of elementary properties of w:

- $\mathrm{d}(z, \mathbb{T}_z) = 1 - |z| \leqslant 1 - |w| = \mathrm{d}(w, \mathbb{T}_w)$, by the Schwarz lemma.
- By [14, Corollary 1.4], $\mathrm{d}(w, \partial G) \asymp |w'(z)|(1 - |z|) = |w'(z)|\mathrm{d}(z, \mathbb{T})$. Since $z \in \tilde{D}$ if and only if $w \in \tilde{G}$, and $|w'(z)| \asymp 1$ for $z \in \tilde{D}$, then

$$\mathrm{d}(w, \partial G) \asymp \mathrm{d}(z, \mathbb{T}), \quad z \in \tilde{D}. \tag{1.1}$$

Here and in what follows the equivalence relation $A \asymp B$ means that $c_1 \leqslant A/B \leqslant c_2$ for generic positive constants c_i, which depend only on G and E. Similarly,

$$\mathrm{d}(w, E) \asymp \mathrm{d}(z, I), \quad z \in \mathbb{D}. \tag{1.2}$$

Indeed, for $z \in \tilde{D}$, $w \in \tilde{G}$, we have $|w'(z)|, |z'(w)| \asymp 1$. For $z \in \mathbb{D}\backslash\tilde{D}$ both sides in (1.2) are equivalent to 1.

Let now $f \in \mathcal{A}(G)$, $|f(0)| = 1$, and assume that for some $p', q', s \geqslant 0$

$$\log\bigl|f(w)\bigr| \leqslant \frac{K|w|^s}{\mathrm{d}^{p'}(w, \mathbb{T})\mathrm{d}^{q'}(w, E)}, \quad w \in G. \tag{1.3}$$

Consider a function $F(z) = f(w(z)) \in \mathcal{A}(\mathbb{D}_z)$. By using the first property of w, equivalence $|w| \asymp |z|$, and (1.2), we obtain

$$\log\bigl|F(z)\bigr| \leqslant \frac{K|z|^s}{\mathrm{d}^{p'}(z, \mathbb{T})\mathrm{d}^{q'}(z, I)}, \quad z \in \mathbb{D}.$$

Theorem 1.1 now implies

$$\sum_{z \in Z_F} \frac{\mathrm{d}^{p'+1+\varepsilon}(z, \mathbb{T})}{|z|^{(s-1+\varepsilon)_+}} \mathrm{d}^{(q'-1+\varepsilon)_+}(z, I) \leqslant C \cdot K$$

for any $0 < \varepsilon < 1$, and, by far,

$$\sum_{z \in \tilde{D} \cap Z_F} \frac{\mathrm{d}^{p'+1+\varepsilon}(z, \mathbb{T})}{|z|^{(s-1+\varepsilon)_+}} \mathrm{d}^{(q'-1+\varepsilon)_+}(z, I) \leqslant C \cdot K.$$

Of course, $Z_f = w(Z_F)$, so by (1.1) and (1.2)

$$\sum_{w\in\tilde{G}\cap Z_F} \frac{\mathrm{d}^{p'+1+\varepsilon}(w,\partial G)}{|w|^{(s-1+\varepsilon)_+}}\mathrm{d}^{(q'-1+\varepsilon)_+}(w,E)\leqslant C\cdot K.$$

Let us show that $\mathrm{d}(w,\partial G)\geqslant C\mathrm{d}(w,\mathbb{T}_w)$, as long as $w\in\tilde{G}$. Indeed, if $\mathrm{d}(w,\partial G)=\mathrm{d}(w,\Delta_2)$, then $\mathrm{d}(w,\partial G)\geqslant \mathrm{d}(w,\mathbb{T}_w)$. Otherwise, $\mathrm{d}(w,\partial G)=\mathrm{d}(w,\Delta_1)$, so $\mathrm{d}(w,\partial G)\geqslant \mathrm{d}'$ and

$$\mathrm{d}(w,\mathbb{T}_w)=1-|w|\leqslant 1\leqslant \frac{\mathrm{d}(w,\partial G)}{\mathrm{d}'},$$

as claimed. Hence

$$\sum_{w\in\tilde{G}\cap Z_F} \frac{\mathrm{d}^{p'+1+\varepsilon}(w,\mathbb{T}_w)}{|w|^{(s-1+\varepsilon)_+}}\mathrm{d}^{(q'-1+\varepsilon)_+}(w,E)\leqslant C\cdot K. \tag{1.4}$$

That is, we have proven

Theorem 1.2. *Let $f\in\mathcal{A}(G)$, $|f(0)|=1$, and for $p',q',s\geqslant 0$*

$$\log\bigl|f(w)\bigr|\leqslant \frac{K|w|^s}{\mathrm{d}^{p'}(w,\mathbb{T})\mathrm{d}^{q'}(w,E)},\quad w\in G.$$

Then (1.4) *holds for any* $0<\varepsilon<1$.

It goes without saying that the similar counterpart of Theorem 0.3 from [1] is also valid in the present setting.

2. Uniformization, Fuchsian groups, and all that

In this section we are aimed at proving Theorem 0.1 with the help of Theorem 1.2.

We start reminding the celebrated uniformization theorem of Klein–Koebe–Poincaré [11, Chapter III], which is one of the key ingredients of the proof. The result is valid for arbitrary Riemann surfaces, but we will formulate it for the so called planar domains, since this is enough for our purposes. Recall that a discrete group of Möbius transformations Γ (of $\mathbb{D}$ on itself) is called a Fuchsian group. The discreteness means that any orbit $\{\gamma(z)\}_{\gamma\in\Gamma}$ is a discrete set in the relative topology of $\mathbb{D}$.

Let $\Omega\subset\overline{\mathbb{C}}$ be a domain with the boundary containing more than two points, and $\lambda_0\in\Omega$. The uniformization theorem says that there exists a covering map $\lambda:\mathbb{D}\to\Omega$, which is unique provided the normalization conditions $\lambda(0)=\lambda_0$, $\lambda'(0)>0$ are set. Moreover, the map is automorphic with respect to a certain Fuchsian group Γ, i.e., $\lambda\circ\gamma=\lambda$ for any $\gamma\in\Gamma$. Symbolically, we write

$$\Omega\simeq\mathbb{D}/\Gamma,$$

where two points $z,w\in\mathbb{D}$ are equivalent with respect to Γ if and only if there is a $\gamma\in\Gamma$ such that $w=\gamma(z)$. For further terminology on the subject, we refer to [11, Chapter III], [12]; see also Simon [13] for a recent presentation.

We will focus upon the special case $\Omega=\overline{\mathbb{C}}\backslash\mathfrak{e}$, defined in (0.2). The standard normalization now is

$$\lambda(0)=\infty,\qquad \lim_{w\to 0} w\lambda(w)=\kappa(\mathfrak{e})>0. \tag{2.1}$$

The properties of the Fuchsian group Γ in this situation are well studied, see [13, Chapter 9.6]. In particular, Γ is a free nonabelian group with n generators $\{\gamma_j\}_{j=1}^n$. The fundamental domain $\mathcal{F}$ (more precisely, its interior $\mathcal{F}^{int}$) is a circular polygon in $\mathbb{D}$, its topological boundary in $\mathbb{D}$ consists of n orthocircles in $\mathbb{C}_+$ and their complex conjugates, and there is a finite distance in $\overline{\mathbb{D}}$ between the different orthocircles, see Fig. 2. We label the vertices of $\mathcal{F}$ by $E=\lambda^{-1}(e)=\{w_j\}$.

The following relations for the covering map are crucial in the sequel.

Lemma 2.1. *Let $w\in\overline{\mathcal{F}}$, closure in $\overline{\mathbb{D}}$, and $\lambda=\lambda(w)$. Then*

$$\mathrm{d}(\lambda,e)\asymp\frac{\mathrm{d}^2(w,E)}{|w|} \tag{2.2}$$

and

$$\mathrm{d}(\lambda,\mathfrak{e})\asymp\frac{\mathrm{d}(w,\mathbb{T}_w)\mathrm{d}(w,E)}{|w|}. \tag{2.3}$$

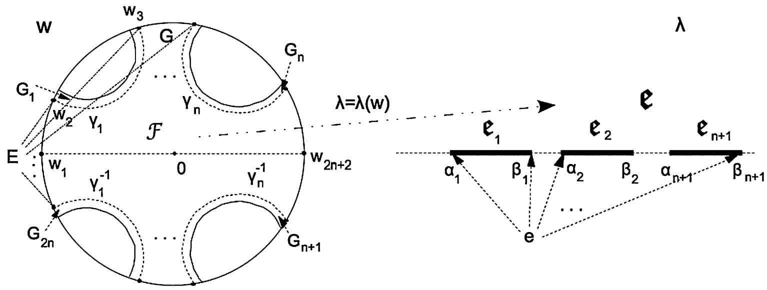

Fig. 2. Uniformization of the domain Ω and the map λ.

Proof. In the case $w \in B(0,r)$ with $0<r<1$ both (2.2) and (2.3) are obvious, since

$$\mathrm{d}(\lambda,e) \asymp \mathrm{d}(\lambda,\mathfrak{e}) \asymp |\lambda| \asymp \frac{1}{|w|}, \qquad \mathrm{d}(w,\mathbb{T}_w) \asymp \mathrm{d}(w,E) \asymp 1$$

by (2.1). So we assume $|w| \geqslant r$.

Put

$$B_j := B_w(w_j,r) \cap \mathcal{F}^{int}, \qquad B := \bigcup B_j,$$

with small enough $r = r(\mathfrak{e})$, so B_j are disjoint. The argument is based on the properties of the covering map (cf., e.g., [13, Theorem 9.6.4]):

1. λ can be extended analytically to a certain domain, which contains $\overline{\mathcal{F}^{int}}$;
2. λ is one–one in $\mathcal{F}^{int}$, and $\lambda'(w)=0$ if and only if $w=w_j$;
3. for $w \in B_j$, we have

$$\lambda(w) = \lambda(w_j) + C_j(w-w_j)^2 + O\big((w-w_j)^3\big), \tag{2.4}$$

and $C_j \neq 0$.

By (2.4), we have for $w \in B_j$

$$\mathrm{d}(\lambda,e) = \big|\lambda(w)-\lambda(w_j)\big| \asymp |w-w_j|^2 = \mathrm{d}(w,E)^2 \asymp \frac{\mathrm{d}(w,E)^2}{|w|},$$

so (2.2) is true on B. For $w \in \overline{\mathcal{F}^{int}} \backslash (B \cup B(0,r))$

$$\mathrm{d}(\lambda,e) \asymp \mathrm{d}(w,E) \asymp |w| \asymp 1,$$

and the proof of (2.2) is complete.

To prove (2.3) for $|w| \geqslant r$ we begin with its simple half

$$\mathrm{d}(\lambda,\mathfrak{e}) \leqslant C\mathrm{d}(w,E)\mathrm{d}(w,\mathbb{T}_w). \tag{2.5}$$

For $w \in B_j$ take $\zeta \in \mathbb{T}_{\mathcal{F}} = \mathbb{T} \cap \overline{\mathcal{F}}$ so that $|w-\zeta| = \mathrm{d}(w,\mathbb{T}_{\mathcal{F}})$. By (2.4)

$$\big|\lambda(\zeta)-\lambda(w)\big| \leqslant \max_{z\in[w,\zeta]} \big|\lambda'(z)\big||\zeta-w| \leqslant C|w-w_j||\zeta-w| = C\mathrm{d}(w,E)\mathrm{d}(w,\mathbb{T}_{\mathcal{F}}).$$

Since $|\lambda(\zeta)-\lambda(w)| \geqslant \mathrm{d}(\lambda,\mathfrak{e})$ and $\mathrm{d}(w,\mathbb{T}_w) \asymp \mathrm{d}(w,\mathbb{T}_{\mathcal{F}})$, (2.5) holds for $w \in B_j$. The similar argument applies in the case $w \in \overline{\mathcal{F}^{int}} \backslash (B \cup B(0,r))$, where $|\lambda'| \asymp 1$, so (2.5) is proved.

Suppose next, that $\mathrm{d}(\lambda,\mathfrak{e}) \geqslant C\mathrm{d}(\lambda,e)$. Then by (2.2) for $|w| \geqslant r$

$$\mathrm{d}(\lambda,\mathfrak{e}) \geqslant C\mathrm{d}^2(w,E) \geqslant C\mathrm{d}(w,E)\mathrm{d}(w,\mathbb{T}_w),$$

which is opposite to (2.5), so (2.3) is true. Hence it remains to consider the case

$$\mathrm{d}(\lambda,\mathfrak{e}) \leqslant \delta\mathrm{d}(\lambda,e), \tag{2.6}$$

δ is small enough.

We apply a version of [14, Corollary 1.4], which reads

$$\mathrm{d}(g,\partial\Omega_2) \asymp \big|g'(w)\big|\mathrm{d}(w,\partial\Omega_1), \tag{2.7}$$

$g:\Omega_1\to\Omega_2$ is a conformal map of bounded domains Ω_j. Let $\Omega_2=B(0,R)\cap\mathbb{C}_-$ be a large semidisk, such that $\mathfrak{e}\subset\partial\Omega_2$, $g=\lambda$ restricted on the preimage of the later set (the part of $\mathcal{F}^{int}$ in the upper half plane away from the origin). The part of (2.6) in $\mathbb{C}_-$ is a union $T=\bigcup T_j$ of small isosceles triangles T_j with bases $\mathfrak{e}_j$. It is clear from the properties of the covering map that

$$\mathrm{d}(\lambda,\partial\Omega_2)=\mathrm{d}(\lambda,\mathfrak{e}),\quad \lambda\in T,$$

$$\mathrm{d}(w,\partial\Omega_1)\asymp\mathrm{d}(w,\mathbb{T}_w),\qquad \left|\lambda'(w)\right|\asymp\mathrm{d}(w,E),\quad w\in\lambda^{(-1)}(T),$$

so by (2.7)

$$\mathrm{d}(\lambda,\mathfrak{e})\asymp\mathrm{d}(w,E)\cdot\mathrm{d}(w,\mathbb{T}_w).$$

The proof is complete. □

Proof of Theorem 0.1. Let $\lambda=\lambda(w):\mathbb{D}_w\to\Omega_\lambda$ be the covering map with normalization (2.1), Γ the corresponding Fuchsian group with generators $\{\gamma_j\}_{j=1}^n$, $E=\lambda^{-1}(e)$ the vertices of $\mathcal{F}$. Put $\gamma_{2n+1-k}:=\gamma_k^{(-1)}$, $k=1,\dots,n$.

Let $f\in\mathcal{A}(\Omega)$ satisfy (0.3). It is clear that $|f(\infty)|=1$. We put $F(w):=f(\lambda(w))$. Then $F\in\mathcal{A}(\mathbb{D})$ and automorphic with respect to Γ. By Lemma 2.1

$$\log\left|F(w)\right|\leqslant\frac{K_1|w|^{p+q}}{\mathrm{d}^p(w,\mathbb{T})\mathrm{d}^{p+2q}(w,E)},\quad w\in\mathcal{F}.\tag{2.8}$$

The special structure of Γ and $\mathcal{F}$ enables one to "inflate" the domain $\mathcal{F}^{int}$ slightly to get another polygon G, so that

$$\mathcal{F}\subset G\subset\mathcal{F}\cup\left(\bigcup_{j=1}^{2n}\gamma_j(\mathcal{F})\right),\qquad \gamma_{n+k}(\mathcal{F})=\overline{\gamma_k(\mathcal{F})},\quad k=1,\dots,n.$$

The distance between the corresponding inner sides of G and $\mathcal{F}^{int}$ is strictly positive.

It is not hard to see that bound (2.8) actually holds in the bigger polygon G. Indeed, let $G_j\subset G\backslash\mathcal{F}^{int}$ be an "annular segment" between the corresponding inner sides of G and $\mathcal{F}^{int}$, so $G\backslash\mathcal{F}^{int}=\bigcup_{j=1}^{2n}G_j$. We have to check (2.8) on each G_j. For $w\in G_j$ there is a unique $z\in\mathcal{F}^{int}$ so that $w=\gamma_j(z)$. Since

$$\mathrm{d}(w,\mathbb{T})=\mathrm{d}\bigl(\gamma_j(z),\gamma_j(\mathbb{T})\bigr)\asymp d(z,\mathbb{T}),$$
$$\mathrm{d}(z,E)=\mathrm{d}\bigl(\gamma_j^{-1}(w),E\bigr)\asymp\mathrm{d}\bigl(w,\gamma_j(E)\bigr)\geqslant C\mathrm{d}(w,E),$$

where we used in an essential way that the number of generators is finite, we see that for $w\in G_j$

$$\log\left|F(w)\right|=\log\left|F(z)\right|\leqslant\frac{K_1|z|^{p+q}}{\mathrm{d}^p(z,\mathbb{T})\mathrm{d}^{p+2q}(z,E)}\leqslant\frac{CK_1|w|^{p+q}}{\mathrm{d}^p(w,\mathbb{T})\mathrm{d}^{p+2q}(w,E)},$$

the first equality being exactly the automorphic property of F. Theorem 1.2 with $s=p+q$ then yields

$$\sum_{w\in\tilde G\cap Z_F}\frac{\mathrm{d}^{p+1+\varepsilon}(w,\mathbb{T}_w)}{|w|^{(p+q-1+\varepsilon)_+}}\mathrm{d}^{(p+2q-1+\varepsilon)_+}(w,E)\leqslant C\cdot K_1\tag{2.9}$$

for $0<\varepsilon<1$, where $\tilde G$ is another polygon with $\mathcal{F}^{int}\subset\tilde G\subset G$. The more so, the same inequality holds for $w\in\overline{\mathcal{F}^{int}}\cap Z_F$.

It remains only to go back to $f\in\mathcal{A}(\Omega)$ and its zero set Z_f. Note that although each point from Z_f has infinitely many preimages in $\mathbb{D}$, we can restrict ourselves with those in $\overline{\mathcal{F}^{int}}$. It follows easily from the properties of the covering map (see the proof of Lemma 2.1) that $1+|\lambda|\asymp\frac{1}{|w|}$. Hence, (2.2) yields

$$\mathrm{d}(w,E)\asymp\left(\frac{\mathrm{d}(\lambda,e)}{1+|\lambda|}\right)^{1/2},$$

and, with the help of (2.3)

$$\mathrm{d}(w,\mathbb{T}_w)\asymp\frac{\mathrm{d}(\lambda,\mathfrak{e})}{(\mathrm{d}(\lambda,e)(1+|\lambda|))^{1/2}}.$$

Substitution of the above relations in (2.9) gives (0.4), and the proof of Theorem 0.1 is complete. □

3. Applications to complex perturbations of a finite-band selfadjoint operator

Consider a bounded finite-band selfadjoint operator A_0, defined on H. Let $A = A_0 + B$, $B \in \mathcal{S}_p$, with $p \geqslant 1$, B is not supposed to be selfadjoint.

The Schatten classes $\mathcal{S}_p$ form a nested family of operator ideals, that is,

1. if $p < q$, then $\mathcal{S}_p \subset \mathcal{S}_q$ and $\|\cdot\|_{\mathcal{S}_q} \leqslant \|\cdot\|_{\mathcal{S}_p}$;
2. if P is a bounded operator, and $Q \in \mathcal{S}_p$, then $PQ, QP \in \mathcal{S}_p$ and $\|PQ\|_{\mathcal{S}_p}, \|QP\|_{\mathcal{S}_p} \leqslant \|P\| \|Q\|_{\mathcal{S}_p}$.

More information on the classes $\mathcal{S}_p$ can be found in monographs [15] and [16].

Given $p \geqslant 1$ put $\lceil p \rceil := \min\{j \in \mathbb{N}: \ j \geqslant p\}$. The following object known as a *regularized perturbation determinant*

$$g_p(\lambda) := \det{}_{\lceil p \rceil}(A - \lambda)(A_0 - \lambda)^{-1}$$

is well defined, $g_p \in \mathcal{A}(\Omega)$, $\Omega = \overline{\mathbb{C}} \backslash \sigma(A_0)$. The basic property of g_p relates its zero set and the discrete spectrum of A:

$$\lambda \in Z_{g_p} \text{ with order } k \text{ if and only if } \lambda \in \sigma_d(A) \text{ with algebraic multiplicity } k.$$

Furthermore, for $\lambda \in \Omega$ the bound

$$\log\left|g_p(\lambda)\right| \leqslant C_p \left\|(A_0 - \lambda)^{-1}\right\|^p \|B\|_{\mathcal{S}_p}^p$$

holds (see, e.g., [16]). For the selfadjoint and finite-band operator A_0 the latter turns into

$$\log\left|g_p(\lambda)\right| \leqslant C_p \frac{\|B\|_{\mathcal{S}_p}^p}{\mathrm{d}^p(\lambda, \mathfrak{e})},$$

which is exactly (0.6).

Theorem 0.4 thus follows directly from Corollary 0.3.

Acknowledgments

L.G. gratefully acknowledges the financial support by the program "Coopération avec les pays de l'Europe de l'Est 2010–2011" of the University of Bordeaux 1. He also wishes to thank the Institute of Mathematics of Bordeaux (IMB, UMR 5251), where the paper was prepared, for the warm hospitality.

References

[1] A. Borichev, L. Golinskii, S. Kupin, A Blaschke-type condition and its application to complex Jacobi matrices, Bull. Lond. Math. Soc. 41 (1) (2009) 117–123.
[2] T. Kato, Variation of discrete spectra, Comm. Math. Phys. 111 (1987) 501–504.
[3] D. Hundertmark, B. Simon, Eigenvalue bounds in the gaps of Schrödinger operators and Jacobi matrices, J. Math. Anal. Appl. 340 (2) (2008) 892–900.
[4] D. Damanik, R. Killip, B. Simon, Perturbations of orthogonal polynomials with periodic recursion coefficients, Ann. of Math. (2) 171 (3) (2010) 1931–2010.
[5] I. Egorova, L. Golinskii, On limit sets for discrete spectrum of complex Jacobi matrices, Mat. Sb. 196 (2005) 43–70.
[6] I. Egorova, L. Golinskii, On the location of the discrete spectrum for complex Jacobi matrices, Proc. Amer. Math. Soc. 133 (12) (2005) 3635–3641.
[7] L. Golinskii, S. Kupin, Lieb–Thirring bounds for complex Jacobi matrices, Lett. Math. Phys. 82 (1) (2007) 79–90.
[8] M. Hansmann, G. Katriel, Inequalities for the eigenvalues of non-selfadjoint Jacobi operators, Complex Anal. Oper. Theory 5 (1) (2011) 197–218.
[9] M. Hansmann, An eigenvalue estimate and its application to non-selfadjoint Jacobi and Schrödinger operators, Lett. Math. Phys. 98 (1) (2011) 79–95.
[10] S. Favorov, L. Golinskii, Blaschke-type conditions for analytic functions in the unit disk: inverse problems and local analogs, preprint, arXiv:1007.3020.
[11] L. Ahlfors, L. Sario, Riemann Surfaces, Princeton University Press, Princeton, NJ, 1960.
[12] L. Ford, Automorphic Functions, McGraw–Hill, 1929.
[13] B. Simon, Szegő's Theorem and Its Descendants. Spectral Theory for L^2 Perturbations of Orthogonal Polynomials, Princeton University Press, Princeton, NJ, 2011.
[14] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Grundlehren Math. Wiss., vol. 299, Springer-Verlag, Berlin, 1992.
[15] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monogr., vol. 18, AMS, Providence, RI, 1969.
[16] B. Simon, Trace Ideals and Their Applications, Math. Surveys Monogr., vol. 120, AMS, Providence, RI, 2005.